In[1]:= 2 - 2

Out[1]= 0

# Biorthogonality & Generalized Spectral Decomposition

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Here I illustrate a technique that permits one to display any finite-dimensional real or complex square matrix as a weighted sum of orthogonal projection matrices. My command of the method has evolved—always in the context provided by specific applications—over the past 25 years. When I look to my record of those occasions I find my remarks to have been often more diffuse/complicated than need be, often colored by details peculiar to the application. My objective here is to illustrate its simple essentials.

## Construct an arbitrary 3×3 complex matrix

In[2]:=
```
𝓜 = Table[RandomComplex[], {i, 1, 3}, {j, 1, 3}];
```

$$\mathbb{M} = \begin{pmatrix} \mathcal{M}[[1]][[1]] & \mathcal{M}[[1]][[2]] & \mathcal{M}[[1]][[3]] \\ \mathcal{M}[[2]][[1]] & \mathcal{M}[[2]][[2]] & \mathcal{M}[[2]][[3]] \\ \mathcal{M}[[3]][[1]] & \mathcal{M}[[3]][[2]] & \mathcal{M}[[3]][[3]] \end{pmatrix};$$

```
M // MatrixForm
```

Out[4]//MatrixForm=

$$\begin{pmatrix} 0.168345 + 0.73991\,i & 0.449493 + 0.603618\,i & 0.562813 + 0.526371\,i \\ 0.786571 + 0.0920774\,i & 0.624823 + 0.32619\,i & 0.392544 + 0.0974287\,i \\ 0.0397631 + 0.171889\,i & 0.112076 + 0.610737\,i & 0.907805 + 0.0989163\,i \end{pmatrix}$$

The matrix happens to be non-singular

In[5]:= Det[M]

Out[5]= $-0.40912 + 0.200337\,i$

but has otherwise no distinguishing special features.

## Construct eigenvalues

In[6]:=
```
Ω = Eigenvalues[M];
ω1 = Ω[[1]];
ω2 = Ω[[2]];
ω3 = Ω[[3]];
Table[ωk, {k, 1, 3}]
```

Out[10]= $\{1.37733 + 1.1002\,i,\ 0.607149 - 0.218146\,i,\ -0.283507 + 0.282958\,i\}$

## Display right/left eigenvectors as column vectors

In[11]:=
```
ℛ = Eigenvectors[𝕄];
Table[a_k = Transpose[{ℛ⟦k⟧}], {k, 1, 3}];
Table[MatrixForm[a_k], {k, 1, 3}]
```

Out[13]=

$$\left\{\begin{pmatrix} 0.640095 + 0.\, i \\ 0.468099 - 0.423426\, i \\ 0.414468 - 0.141731\, i \end{pmatrix}, \begin{pmatrix} -0.19978 + 0.212769\, i \\ -0.428152 + 0.251036\, i \\ 0.817609 + 0.\, i \end{pmatrix}, \begin{pmatrix} 0.753228 + 0.\, i \\ -0.608007 - 0.12896\, i \\ -0.0657228 + 0.204999\, i \end{pmatrix}\right\}$$

In[14]:=
```
ℒ = Eigenvectors[Transpose[𝕄]];
Table[b_k = Transpose[{ℒ⟦k⟧}], {k, 1, 3}];
Table[MatrixForm[b_k], {k, 1, 3}]
```

Out[16]=

$$\left\{\begin{pmatrix} 0.483656 - 0.0399091\, i \\ 0.664422 + 0.\, i \\ 0.455728 - 0.339617\, i \end{pmatrix}, \begin{pmatrix} -0.298082 - 0.195574\, i \\ -0.431238 + 0.12541\, i \\ 0.81927 + 0.\, i \end{pmatrix}, \begin{pmatrix} 0.749193 + 0.\, i \\ -0.505316 - 0.325768\, i \\ -0.181575 - 0.210407\, i \end{pmatrix}\right\}$$

From the following hermitian matrices

In[17]:=
```
Table[Chop[Conjugate[Transpose[a_i]].a_j]⟦1⟧⟦1⟧, {i, 1, 3}, {j, 1, 3}] // MatrixForm
```

Out[17]//MatrixForm=

$$\begin{pmatrix} 1. & -0.095718 + 0.188292\, i & 0.195841 - 0.242161\, i \\ -0.095718 - 0.188292\, i & 1. & 0.0237299 + 0.215191\, i \\ 0.195841 + 0.242161\, i & 0.0237299 - 0.215191\, i & 1. \end{pmatrix}$$

In[18]:=
```
Table[Chop[Conjugate[Transpose[b_i]].b_j]⟦1⟧⟦1⟧, {i, 1, 3}, {j, 1, 3}] // MatrixForm
```

Out[18]//MatrixForm=

$$\begin{pmatrix} 1. & -0.0495244 + 0.255076\, i & 0.0153179 - 0.344102\, i \\ -0.0495244 - 0.255076\, i & 1. & -0.195023 + 0.177998\, i \\ 0.0153179 + 0.344102\, i & -0.195023 - 0.177998\, i & 1. \end{pmatrix}$$

we see that the right (ditto the left) eigenvectors, as supplied by Mathematica, have been automatically normalized—inessential for the present argument—and are not orthogonal (as they would be if 𝕄 were hermitian, which in the most commonly encountered instances of spectral decomposition is assumed: our objective is to **dispense with that assumption**).

From—NOTE the use here and below of `Transpose[●]` where one might expect to see `Conjugate[Transpose[●]]`—

In[19]:=
```
Table[Chop[Transpose[b_i].a_j]⟦1⟧⟦1⟧, {i, 1, 3}, {j, 1, 3}] // MatrixForm
```

Out[19]//MatrixForm=

$$\begin{pmatrix} 0.761352 - 0.51223\, i & 0 & 0 \\ 0 & 0.924159 - 0.186302\, i & 0 \\ 0 & 0 & 0.884605 + 0.23984\, i \end{pmatrix}$$

we see that $b_1$ is normal to $a_2$ & $a_3$, etc. To replace the diagonal elements with 1s, and thus to **achieve biorthogonality**, we rescale the b-vectors:

In[20]:=
```
Table[d_k = Part[Transpose[b_k].a_k, 1, 1], {k, 1, 3}];
Table[A_k = b_k / d_k, {k, 1, 3}];
Table[Chop[Transpose[A_i].a_j][[1]][[1]], {i, 1, 3}, {j, 1, 3}] // MatrixForm
```

Out[22]//MatrixForm=

$$\begin{pmatrix} 1. & 0 & 0 \\ 0 & 1. & 0 \\ 0 & 0 & 1. \end{pmatrix}$$

## Construction of the associated projection matrices

Let matrices $\mathbb{P}_k$ be defined

In[23]:=
```
Table[ℙ_k = a_k.Transpose[A_k], {k, 1, 3}];
```

Those matrices are projective

In[24]:=
```
Table[MatrixForm[Chop[ℙ_k.ℙ_k - ℙ_k]], {k, 1, 3}]
```

Out[24]= $\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\}$

and therefore singular (one cannot “unproject” except in the trivial case $\mathbb{I}$):

In[25]:=
```
Table[Chop[Det[ℙ_k]], {k, 1, 3}]
```

Out[25]= {0, 0, 0}

They are, moreover, orthogonal

In[26]:=
```
MatrixForm[Chop[ℙ_1.ℙ_2]]
MatrixForm[Chop[ℙ_1.ℙ_3]]
MatrixForm[Chop[ℙ_2.ℙ_3]]
```

Out[26]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Out[27]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Out[28]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and complete:

In[29]:=
```
MatrixForm[Chop[ℙ_1 + ℙ_2 + ℙ_3]]
```

Out[29]//MatrixForm=

$$\begin{pmatrix} 1. & 0 & 0 \\ 0 & 1. & 0 \\ 0 & 0 & 1. \end{pmatrix}$$

Finally, they provide this spectral decomposition of $\mathbb{M}$ :

In[30]:= `MatrixForm[Chop[ω1 ℙ1 + ω2 ℙ2 + ω3 ℙ3 - 𝕄]]`

Out[30]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This places one in position to (for example) obtain

In[31]:= `MatrixExp[𝕄] // MatrixForm`

Out[31]//MatrixForm=

$$\begin{pmatrix} 0.60687 + 1.29824\, i & -0.420605 + 1.38691\, i & 0.357269 + 1.57802\, i \\ 0.819978 + 0.940072\, i & 1.54938 + 1.38049\, i & 0.863913 + 1.02162\, i \\ -0.260296 + 0.703322\, i & -0.339543 + 1.37701\, i & 2.15611 + 0.66776\, i \end{pmatrix}$$

from

In[32]:= `MatrixForm[Exp[ω1] ℙ1 + Exp[ω2] ℙ2 + Exp[ω3] ℙ3]`

Out[32]//MatrixForm=

$$\begin{pmatrix} 0.60687 + 1.29824\, i & -0.420605 + 1.38691\, i & 0.357269 + 1.57802\, i \\ 0.819978 + 0.940072\, i & 1.54938 + 1.38049\, i & 0.863913 + 1.02162\, i \\ -0.260296 + 0.703322\, i & -0.339543 + 1.37701\, i & 2.15611 + 0.66776\, i \end{pmatrix}$$

Similarly

In[33]:= `Inverse[𝕄] // MatrixForm`

$$\text{MatrixForm}\left[\frac{1}{\omega_1}\mathbb{P}_1 + \frac{1}{\omega_2}\mathbb{P}_2 + \frac{1}{\omega_3}\mathbb{P}_3\right]$$

Out[33]//MatrixForm=

$$\begin{pmatrix} -0.981699 - 0.742891\, i & 1.01306 + 0.959766\, i & -0.10084 + 0.517039\, i \\ 1.30514 + 0.859189\, i & 0.259675 - 1.26773\, i & -0.635565 - 0.699867\, i \\ 0.233037 - 0.791092\, i & -0.765952 - 0.168587\, i & 0.856939 + 0.417062\, i \end{pmatrix}$$

Out[34]//MatrixForm=

$$\begin{pmatrix} -0.981699 - 0.742891\, i & 1.01306 + 0.959766\, i & -0.10084 + 0.517039\, i \\ 1.30514 + 0.859189\, i & 0.259675 - 1.26773\, i & -0.635565 - 0.699867\, i \\ 0.233037 - 0.791092\, i & -0.765952 - 0.168587\, i & 0.856939 + 0.417062\, i \end{pmatrix}$$

## Remarks:

1. If $\mathbb{M}$ is symmetric then the distinction between left/right eigenvectors evaporates, and the construction reduces to the more familiar spectral decomposition.

2. Spectral degeneracy poses only the familiar difficulty: one has "exercise an option," to deposit—arbitrarily, "by hand"—distinct eigenvectors on the multidimensional eigenspaces. Which Mathematica is happy to do for you.

3. One sometimes has interest in sets of (generally non-orthogonal) vectors that come to one's attention NOT as the eigenvectors of a matrix. Suppose, for example, that
3-vectors $a_1$, $a_2$, $a_3$, defined by the primitive parallelogram. are used to describe a lattice in 3-space. Interest then attaches (in X-ray crystallography, solid state physics) to the Bravais lattice, defined by the vectors $A_1$, $A_2$, $A_3$ biorthogonal to $a_1$, $a_2$, $a_3$. In that context normalization/rescaling (achieved above by constants $d_k$) is achieved by scalar triple produces (see "Reciprocal systems of non-orthogo-

nal quantum states," June 1998). In such contexts "spectral decomposition" plays no role, so far as I am aware (there is no matrix to decompose!).

4. The scheme described above extends to ∞ dimensions with only the familiar adjustments.

5. The $\mathbb{M}$-based biorthogonal basis and associated projection matrices can, by completeness ($\mathbb{I}$ = sum $\mathbb{P}_k$), be used to develop representations of *any* vector
v = $\mathbb{I}$.v and *any* matrix $\mathbb{X} = \mathbb{I}.\mathbb{X}.\mathbb{I}$.

6. The scheme which I have here described by example can easily be described in more general terms on the type-set page. It is so elementary that it would fit comfortably into introductory texts. Perhaps these days it can be found in such places? It's a long time since I have read such a text, and things may have changed.